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# SOME RELATIONS ON FOURIER COEFFICIENTS OF DEGREE 2 SIEGEL FORMS OF ARBITRARY LEVEL

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ABSTRACT. We extend some recent work of D. McCarthy, proving relations among some Fourier coefficients of a degree 2 Siegel modular form  $F$  with arbitrary level and character, provided there are some primes  $p$  so that  $F$  is an eigenform for the Hecke operators  $T(p)$  and  $T_1(p^2)$ .

## 1. INTRODUCTION

In a recent paper [3], McCarthy derives some nice results for Fourier coefficients and Hecke eigenvalues of degree 2 Siegel modular forms of level 1, extending some classical results regarding elliptic modular forms. In particular, with  $F$  a degree 2, level 1 Siegel modular form that is an eigenform for all the Hecke operators  $T(p)$ ,  $T(p^2)$  ( $p$  prime), and  $a(T)$  denoting the  $T$ th Fourier coefficient of  $F$ , McCarthy shows that:

- (a) provided that  $a(I) = 1$  and  $p$  is prime, the  $T(p)$ -eigenvalue  $\lambda(p)$  and the  $T(p^2)$ -eigenvalue  $\lambda(p^2)$  are described explicitly in terms of  $a(pI)$  and  $a(p^2I)$ ;
- (b) for  $r \geq 1$ ,  $a(I)a(p^{r+1}I)$  is described explicitly in terms of  $a(I)$ ,  $a(pI)$ ,  $a(p^{r-1}I)$ ,  $a\left(\begin{smallmatrix} p^{r-1} & \\ & p^{r+1} \end{smallmatrix}\right)$ , and  $a\left(p^r \begin{pmatrix} (1+u^2)/p & u \\ u & p \end{pmatrix}\right)$  where  $1 \leq u < p/2$  with  $u^2 \not\equiv 1 \pmod{p}$ ;
- (c) if  $a(I) = 0$  then  $a(mI) = 0$  for all  $m \in \mathbb{Z}_+$ ; further, if  $m, n \in \mathbb{Z}_+$  with  $(m, n) = 1$ , then  $a(I)a(mnI) = a(mI)a(nI)$ .

(As defined in Sec. 2,  $T_2(p^2)$  is the Hecke operator associated with the matrix  $\text{diag}(p, p, 1/p, 1/p)$ ,  $T_1(p^2)$  is the Hecke operator associated with the matrix  $\text{diag}(p, 1, 1/p, 1)$ , and  $T(p^2) = T_2(p^2) + p^{k-3}T_1(p^2) + p^{2k-6}$ . In [2], for  $\chi = 1, T(p^2)$  is denoted by  $\tilde{T}_2(p^2)$ .) McCarthy's approach begins with some formulas from [1], which are somewhat cumbersome.

In this note we use the formulas from [2] that give the action of Hecke operators on Fourier coefficients of a Siegel modular form  $F$ , allowing for arbitrary level and character, and giving a simpler proof of McCarthy's above results (with no restriction on the level or character). Here when we say that a modular form has weight  $k$ , level  $\mathcal{N}$  and character  $\chi$ , we mean that it transforms with weight  $k$  and character  $\chi$  under the congruence subgroup

$$\Gamma_0(\mathcal{N}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_2(\mathbb{Z}) : \mathcal{N} | C \right\},$$

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where  $Sp_2(\mathbb{Z})$  is the symplectic group of  $4 \times 4$  integral matrices. We work with “Fourier coefficients” attached to lattices (as explained below), making it simpler to work with the image of  $F$  under a Hecke operator. For  $p$  prime and degree 2, the local Hecke algebra is generated by  $T(p)$ ,  $T_1(p^2)$  and  $T_2(p^2)$ . When  $\mathcal{N} = 1$ , Proposition 5.1 of [2] gives a relation between these generators, from which we deduce that with  $p \nmid \mathcal{N}$ ,  $T(p)$  and  $T_1(p^2)$  generate the local Hecke algebra, as do  $T(p)$  and  $\tilde{T}_2(p^2)$ . However, when  $p|\mathcal{N}$ , we have  $T_2(p^2) = (T(p))^2$ . Hence in this note we use the local generators  $T(p)$  and  $T_1(p^2)$ ; to more easily apply the results of [2], we use the operator

$$\tilde{T}_1(p^2) = T_1(p^2) + \chi(p)p^{k-3}(p+1)$$

in place of  $T_1(p^2)$ .

Using some rather special aspects of working with degree 2 Siegel modular forms, we prove the following extensions of [3].

**Theorem 1.1.** *Suppose that  $F$  is a degree 2 Siegel modular form of weight  $k \in \mathbb{Z}_+$ , level  $\mathcal{N}$  and character  $\chi$  with Fourier expansion*

$$F(\tau) = \sum_T a(T) \exp(2\pi i \text{Tr}(T\tau)).$$

*Also suppose that  $p$  is prime with  $F|T(p) = \lambda(p)F$  and  $F|\tilde{T}_1(p^2) = \tilde{\lambda}_1(p^2)F$ .*

(a) *We have*

$$\lambda(p)a(mI) = \chi(p)p^{k-2}\eta(p)a(mI) + a(mpI),$$

where

$$\eta(p) = \begin{cases} 1 + \chi(-1)(-1)^k & \text{if } p \equiv 1 \pmod{4}, \\ 0 & \text{if } p \equiv 3 \pmod{4}, \\ 1 & \text{if } p = 2. \end{cases}$$

(Thus when  $a(mI) \neq 0$ ,  $\lambda(p)$  is given explicitly in terms of  $p$ ,  $a(mI)$  and  $a(pmI)$ .) As well, we have

$$\begin{aligned} \chi(p)p^{k-2}\tilde{\lambda}_1(p^2)a(mI) &= \chi(p^2)p^{2k-4}(\alpha(I;p) - p)a(mI) \\ &\quad + \lambda(p)a(pmI) - a(p^2mI) \end{aligned}$$

where

$$\alpha(I;p) = \begin{cases} 2 & \text{if } p \equiv 1 \pmod{4}, \\ 0 & \text{if } p \equiv 3 \pmod{4}, \\ 1 & \text{if } p = 2. \end{cases}$$

(Thus when  $\chi(p)a(mI) \neq 0$ ,  $\tilde{\lambda}_1(p^2)$  is given explicitly in terms of  $p$ ,  $a(mI)$ ,  $a(pmI)$  and  $a(p^2mI)$ .)

(b) Set  $\epsilon = 1 + \chi(-1)(-1)^k$ . For  $r \geq 1$ ,  $a(mI)a(p^{r+1}I)$  is given by

$$\begin{aligned} &a(pmI)a(p^rI) - \chi(p^2)p^{2k-3}a(mI)a(p^{r-1}I) \\ &\quad + \epsilon\chi(p)p^{k-2}a(mI)a\left(\begin{smallmatrix} p^{r-1}m \\ p^{r+1}m \end{smallmatrix}\right) \\ &\quad + \epsilon\chi(p)p^{k-2}a(mI) \sum_{\substack{1 \leq u < p/2 \\ u^2 \not\equiv -1 \pmod{p}}} a\left(\begin{smallmatrix} p^r m \begin{pmatrix} (1+u^2)/p & u \\ u & p \end{pmatrix} \end{smallmatrix}\right). \end{aligned}$$

- (c) Suppose that  $n$  a product of powers of primes  $p$  so that  $F$  is an eigenform for  $T(p)$  and  $\tilde{T}_1(p^2)$ , and that  $m \in \mathbb{Z}_+$  with  $(m, n) = 1$ . If  $a(mI) = 0$  then  $a(mnI) = 0$ . Also, we have  $a(I)a(mnI) = a(mI)a(nI)$ .

We also prove the following modest generalization.

**Theorem 1.2.** Suppose that  $F$  is a degree 2 Siegel modular form of weight  $k \in \mathbb{Z}_+$ , level  $\mathcal{N}$  and character  $\chi$  with Fourier expansion

$$F(\tau) = \sum_T a(T) \exp(2\pi i \text{Tr}(T\tau)).$$

Suppose that  $p$  is an odd prime, and set  $D = \begin{pmatrix} 1 & \\ & p \end{pmatrix}$ . Let  $\mathcal{S}$  be the set of odd primes so that for  $q \in \mathcal{S}$ ,  $F$  is an eigenform for  $T(q)$  and  $\tilde{T}_1(q^2)$ , and either  $q = p$  or  $\left(\frac{-p}{q}\right) = -1$ . Let  $n$  be a product of powers of primes in  $\mathcal{S}$ . Then for any  $m \in \mathbb{Z}_+$  so that  $(m, n) = 1$ , we have

$$a(D)a(mnD) = a(mD)a(nD).$$

Also,  $a(D)a(mnD) = 0$  if  $a(mD) = 0$ .

We note that McCarthy applies his results to compute eigenvalues of the level 1 Eisenstein series with regard to the Hecke operators  $T(p^r)$  ( $p$  prime); as he notes, in [5] we computed the Hecke-eigenvalues of Eisenstein series of square-free levels for all primes  $p$ , allowing nontrivial character (then generalized in [6] for arbitrary level  $\mathcal{N}$  and character  $\chi$ , but only for primes  $p$  so that  $p^2 \nmid \mathcal{N}$ ).

We further note that it seems that these results cannot be extended to higher degrees, as Lemma 3.1 (which is pivotal for our arguments) does not extend to higher degrees.

## 2. PRELIMINARIES

We will use some language and notation commonly used in quadratic forms and modular forms theory. When  $\Lambda$  is a lattice whose quadratic form is given by the matrix  $T$  (relative to some  $\mathbb{Z}$ -basis for  $\Lambda$ ), we write  $\Lambda \simeq T$ . Now suppose that  $\Lambda$  is a lattice with  $\Lambda \simeq T$  and that  $m \in \mathbb{Q}_+$ ; we write  $\Lambda^m$  to denote the lattice  $\Lambda$  “scaled” by  $m$ , meaning that  $\Lambda^m \simeq mT$ . Also, the discriminant of  $\Lambda$  is  $\det T$ . With  $\Lambda, \Omega$  lattices on the same underlying quadratic space over  $\mathbb{Q}$ , we write  $\{\Lambda : \Omega\}$  to denote the invariant factors of  $\Omega$  in  $\Lambda$ .

We set

$$\mathfrak{h}_{(2)} = \{X + iY : X, Y \in \mathbb{R}_{\text{sym}}^{2,2} : Y > 0\},$$

where  $\mathbb{R}_{\text{sym}}^{2,2}$  denotes the set of  $2 \times 2$  symmetric matrices with real entries, and  $Y > 0$  means that  $Y$  represents a positive definite quadratic form. For a ring  $R$ , we write  $Sp_2(R)$  for the group of  $4 \times 4$  symplectic matrices with entries in  $R$ . Fixing a weight  $k \in \mathbb{Z}_+$ , for  $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_2(\mathbb{Q})$ , we define

$$F(\tau)|\gamma = (\det \gamma)^{k/2} \det(C\tau + D)^{-k} F((A\tau + B)(C\tau + D)^{-1}).$$

When  $F$  is a degree 2 Siegel modular form of weight  $k$ , level  $\mathcal{N}$  and character  $\chi$ , this means that for  $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(\mathcal{N})$ , we have

$$F(\tau)|\gamma = \chi(\det D_\gamma)F(\tau).$$

We can write  $F$  as a Fourier series:

$$F(\tau) = \sum_{T \geq 0} a(T) \exp(2\pi i \text{Tr}(T\tau))$$

where the sum is over  $2 \times 2$  symmetric, positive semi-definite, half-integral matrices  $T$  (so the entries in  $T$  are half-integers with integers on the diagonal). Given  $G \in GL_2(\mathbb{Z})$ , we have  $\gamma = \begin{pmatrix} G^{-1} & \\ & {}^tG \end{pmatrix} \in \Gamma_0(\mathcal{N})$ . Hence

$$\begin{aligned} \chi(\det G)F(\tau) &= F(\tau)|\gamma \\ &= (\det G)^k F(G^{-1}\tau {}^tG^{-1}) \\ &= (\det G)^k \sum_T a({}^tGTG) \exp(2\pi i \text{Tr}(T\tau)). \end{aligned}$$

Thus  $a({}^tGTG) = \chi(\det G)(\det G)^k a(T)$ . So we can also write  $F$  as a “Fourier series” supported on isometry classes of even integral, positive semi-definite lattices: For  $\Lambda$  an even integral lattice with  $\mathbb{Z}$ -basis  $\{x, y\}$ , set  $c(\Lambda) = a(T_\Lambda)$  where, relative to the given basis for  $\Lambda$ , we have  $\Lambda \simeq 2T_\Lambda$ . When  $\chi(-1) \neq (-1)^k$ , we equip  $\Lambda$  with an orientation, meaning that with  $G \in GL_2(\mathbb{Z})$ ,  $(x \ y)G$  is a basis for the oriented lattice  $\Lambda$  if and only if  $\det G = 1$ . Then

$$F(\tau) = \sum_{\text{cls } \Lambda} c(\Lambda) e^*\{\Lambda\tau\}$$

where  $\text{cls } \Lambda$  varies over all isometry classes of (oriented) even integral, positive semi-definite lattices, and

$$e^*\{\Lambda\tau\} = \sum_G \exp(2\pi i \text{Tr}({}^tGT_\Lambda G\tau))$$

where  $G$  varies over  $O(\Lambda) \backslash GL_2(\mathbb{Z})$  when  $\chi(-1) = (-1)^k$ , and  $G$  varies over  $O^+(\Lambda) \backslash SL_2(\mathbb{Z})$  otherwise. (Here  $O(\Lambda)$  denotes the orthogonal group of  $\Lambda$ , and  $O^+(\Lambda) = O(\Lambda) \cap SL_2(\mathbb{Z})$ .)

Still suppose that  $F$  is a Siegel modular form of degree 2, weight  $k$ , level  $\mathcal{N}$  and character  $\chi$ . For  $p$  prime, we define  $T(p)$ ,  $T_1(p^2)$ , and  $T_2(p^2)$  as follows. Take  $\delta(p) = \text{diag}(p, p, 1, 1)$ ,  $\delta_1(p^2) = (p, 1, 1/p, 1)$ , and  $\delta_2(p^2) = \text{diag}(p, p, 1/p, 1/p)$ . With  $\Gamma = \Gamma_0(\mathcal{N})$ , we set

$$F|T(p) = p^{k-3} \sum_{\gamma} \bar{\chi}(\gamma) F|\delta(p)^{-1}\gamma$$

where  $\gamma$  varies over  $(\delta(p)\Gamma\delta(p)^{-1} \cap \Gamma) \backslash \Gamma$ , and for  $j = 1, 2$ , we set

$$F|T_j(p^2) = p^{j(k-3)} \sum_{\gamma} \bar{\chi}(\gamma) F|\delta_j(p^2)^{-1}\gamma$$

where  $\gamma$  varies over  $(\delta_j(p^2)\Gamma\delta_j(p^2)^{-1} \cap \Gamma) \backslash \Gamma$ . Note that replacing  $\delta(p)$  or  $\delta_j(p^2)$  by a scalar multiple of itself does not change the definition of the associated Hecke operator. Note also that in [2], we did not normalize

$T_j(p^2)$  by  $p^{j(k-3)}$ , as is usually done in other texts, and has been done in the above formula for  $T_1(p^2)$ . With  $\tilde{T}_1(p^2) = T_1(p^2) + \chi(p)p^{k-3}(p+1)$ , Theorem 6.1 of [2] gives us the following.

**Theorem 2.1.** *Let  $F$  be a degree 2 Siegel modular form of weight  $k$ , level  $\mathcal{N}$ , character  $\chi$ , and lattice coefficients  $c(\Lambda)$ . Then for any even integral lattice  $\Lambda$ , the  $\Lambda$ th coefficient of  $F|T(p)$  is*

$$\chi(p^2)p^{2k-3}c(\Lambda^{1/p}) + \chi(p)p^{k-2} \cdot \sum_{\{\Lambda:\Omega\}=(1,p)} c(\Omega^{1/p}) + c(\Lambda^p),$$

and the  $\Lambda$ th coefficient of  $F|\tilde{T}_1(p^2)$  is

$$\chi(p^2)p^{2k-3} \cdot \sum_{\{\Lambda:\Omega\}=(1/p,1)} c(\Omega) + \chi(p)p^{k-2}\alpha(\Lambda;p)c(\Lambda) + \sum_{\{\Lambda:\Omega\}=(1,p)} c(\Omega).$$

With  $Q$  the quadratic form on  $\Lambda$ , we equip  $\Lambda/p\Lambda$  with the quadratic form  $\frac{1}{2}Q$ , and  $\alpha(\Lambda;p)$  is the number of isotropic lines in the quadratic space  $\Lambda/p\Lambda$ . There are  $p+1$  lines in  $\Lambda/p\Lambda$ , and each of these lines is generated either by  $y + p\Lambda$  or by  $(x + uy) + p\Lambda$  for some  $u$  with  $0 \leq u < p$ . So with  $\Lambda \simeq 2I$ ,  $\alpha(\Lambda;2) = 1$ ,  $\alpha(\Lambda;p) = 2$  when  $p \equiv 1 \pmod{4}$ , and  $\alpha(\Lambda;p) = 0$  when  $p \equiv 3 \pmod{4}$ . When  $\Lambda \simeq 2T$  with  $p|T$ ,  $\alpha(\Lambda;p) = p+1$ .

Note that with  $p$  a prime and  $m \in \mathbb{Z}_+$  so that  $p \nmid m$ , for any even integral rank 2 lattice  $\Lambda$  we have  $\alpha(\Lambda;p) = \alpha(\Lambda^m;p)$  since scaling by  $m$  does not change whether a line is isotropic in  $\Lambda/p\Lambda$ .

### 3. PROOF OF THEOREM 1.1

The next lemma is pivotal in our proof of Theorem 1.1; when this lemma generalizes, we can generalize this theorem (as seen in Theorem 1.2).

**Lemma 3.1.** *Suppose that  $F$  is a degree 2 Siegel modular form of weight  $k$ , level  $\mathcal{N}$ , character  $\chi$ , and lattice coefficients  $c(\Lambda)$ . With  $\Delta \simeq 2I$ ,  $p$  prime and  $m \in \mathbb{Z}_+$  so that  $p \nmid m$ , we have*

$$\sum_{\{\Delta:\Omega\}=(1/p,1)} c(\Omega^{pm}) = \sum_{\{\Delta:\Omega\}=(1,p)} c(\Omega^{m/p}) = \eta(p)c(\Delta^m)$$

where, as in Theorem 1.1,

$$\eta(p) = \begin{cases} 1 + \chi(-1)(-1)^k & \text{if } p \equiv 1 \pmod{4}, \\ 0 & \text{if } p \equiv 3 \pmod{4}, \\ 1 & \text{if } p = 2. \end{cases}$$

*Proof.* Suppose that  $\{\Delta:\Omega\} = (1/p,1)$ . Then  $\{\Delta:p\Omega\} = (1,p)$ ; also, with  $T$  a matrix so that  $\Omega^{m/p} \simeq \frac{m}{p}T$ , we have  $p\Omega^{m/p} \simeq pmT$ . This proves that

$$\sum_{\{\Delta:\Omega\}=(1/p,1)} c(\Omega^{pm}) = \sum_{\{\Delta:\Omega\}=(1,p)} c(\Omega^{m/p}).$$

Let  $\{x, y\}$  be a basis for  $\Delta$  relative to which  $\Delta \simeq 2I$ , and suppose that  $\{\Delta:\Omega\} = (1,p)$ . Thus  $\Omega = \mathbb{Z}(x + uy) \oplus \mathbb{Z}py$  for  $0 \leq u < p$  or  $\Omega = \mathbb{Z}px \oplus \mathbb{Z}y$ . Hence  $\Omega^{m/p}$  is even integral if and only if  $\Omega = \mathbb{Z}(x + uy) \oplus \mathbb{Z}py$  with  $u^2 \equiv -1 \pmod{p}$ . If  $p \equiv 3 \pmod{4}$ , there are no such  $u$ . Suppose that  $p \equiv 1 \pmod{4}$ , and fix  $u$

so that  $u^2 \equiv -1 \pmod{p}$ . Set  $\Omega_u = \mathbb{Z}(x + uy) \oplus \mathbb{Z}py$  and  $\Omega_{-u} = \mathbb{Z}(x - uy) \oplus \mathbb{Z}py$ . Then  $\Omega_u^{1/p}$  and  $\Omega_{-u}^{1/p}$  are integral with determinant 1. Thus by Exercise 5 p. 77 of [4], there is some  $G \in GL_2(\mathbb{Z})$  so that  ${}^tGTG = I$ . Therefore  $c(\Omega_u^{m/p}) = \chi(\det G)(\det G)^k c(\Delta^m)$ . When  $p = 2$ ,  $\Omega^{m/2}$  is even integral only for  $\Omega_1 = \mathbb{Z}(x + y) \oplus \mathbb{Z}2y \simeq 2 \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ . Since  ${}^tG \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} G = I$  for  $G = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ , we have  $c(\Omega_1^{1/2}) = c(\Delta)$ . Thus when  $p = 2$ , the sum on  $\Omega$  is  $c(\Omega^{m/2}) = c(\Delta^m)$ .  $\square$

In the next proposition we use Lemma 3.1 to establish some very useful identities.

**Proposition 3.2.** *Suppose that  $F$  is a degree 2 Siegel modular form of weight  $k$ , level  $\mathcal{N}$ , character  $\chi$ , and lattice coefficients  $c(\Lambda)$ . Also suppose that  $F|T(p) = \lambda(p)F$  and  $F|\tilde{T}_1(p^2) = \tilde{\lambda}_1(p^2)F$ . Set  $\eta(1) = 0$ ,  $\kappa(1) = 1$ . With  $\Delta \simeq 2I$  and  $m \in \mathbb{Z}_+$  so that  $p \nmid m$ , for  $r \geq 1$  we inductively define  $\eta(p^r)$  and  $\kappa(p^r)$  as follows:  $\eta(p)$  is as in Proposition 3.1,  $\kappa(p) = \lambda(p) - \chi(p)p^{k-2}\eta(p)$ , and for  $r \geq 2$ ,*

$$\eta(p^r) = \tilde{\lambda}_1(p^2)\kappa(p^{r-2}) - \chi(p^2)p^{2k-3}\eta(p^{r-2}) - \chi(p)p^{k-2}\alpha(\Delta^{p^{r-2}}; p)\kappa(p^{r-2})$$

and

$$\kappa(p^r) = \lambda(p)\kappa(p^{r-1}) - \chi(p^2)p^{2k-3}\kappa(p^{r-2}) - \chi(p)p^{k-2}\eta(p^r).$$

Then we have

$$(1) \quad \sum_{\{\Delta:\Omega\}=(1/p,1)} c(\Omega^{p^r m}) = \sum_{\{\Delta:\Omega\}=(1,p)} c(\Omega^{p^{r-2}m}) = \eta(p^r)c(\Delta^m)$$

and

$$(2) \quad c(\Delta^{p^r m}) = \kappa(p^r)c(\Delta^m).$$

*Proof.* Recall that the value of  $\alpha(\Delta; p)$  is computed after Theorem 2.1; note that for  $r \geq 1$ ,  $\alpha(\Delta^{p^r}; p) = p + 1$  as then  $\Delta^{p^r}/p\Delta^{p^r}$  is totally isotropic and contains  $p + 1$  lines. Also, note that the first equality in Equation (1) is easily verified by replacing  $\Omega$  by  $p\Omega$ . We now compute  $\eta(p^r)$  and  $\kappa(p^r)$ .

(Case  $r = 0$ ;) With  $\kappa(1) = 1$ , it is clear that  $c(\Delta) = \kappa(1)c(\Delta)$ . So suppose that we have  $\{\Delta : \Omega\} = (1, p)$ . Then  $\text{disc } \Omega^{m/p^2} = 4m^2/p^2$ . Hence when  $p \neq 2$ ,  $\Omega^{m/p}$  cannot be integral, so  $c(\Omega^{m/p}) = 0$ . When  $p = 2$ , we see from the discussion at the end of the proof of Lemma 3.1 that  $\Omega^{m/4}$  is not even integral for any  $\Omega$  with  $\{\Delta : \Omega\} = (1, 2)$ . Thus Equation (1) holds with  $\eta(1) = 0$ .

(Case  $r = 1$ ;) In Lemma 3.1 we showed that Equation (1) holds with  $\eta(p)$  as defined therein. We know that  $c(\Delta^{m/p}) = 0$  since  $\Delta^{m/p}$  is not even integral, and so by Theorem 2.1 and the above conclusion we have

$$\kappa(p)c(\Delta^m)\lambda(p)c(\Delta^m) - \chi(p)p^{k-2}\eta(p)c(\Delta^m).$$

(Induction step;) Suppose that  $r \geq 2$  and that the proposition holds for all  $\ell$  with  $0 \leq \ell < r$ . First, from Theorem 2.1 and the induction hypothesis

we have

$$\begin{aligned} \sum_{\{\Delta:\Omega\}=(1,p)} c(\Omega^{p^{r-2}m}) &= (\tilde{\lambda}_1(p^2)\kappa(p^{r-2}) - \chi(p^2)p^{2k-3}\eta(p^{r-2}))c(\Delta^m) \\ &\quad - \chi(p)p^{k-2}\alpha(\Delta^{p^{r-2}};p)\kappa(p^{r-2})c(\Delta^m) \\ &= \eta(p^r)c(\Delta^m). \end{aligned}$$

Hence we also have

$$\begin{aligned} c(\Delta^{p^r m}) &= (\lambda(p)\kappa(p^{r-1}) - \chi(p^2)p^{2k-3}\kappa(p^{r-2}) - \chi(p)p^{k-2}\eta(p^r))c(\Delta^m) \\ &= \kappa(p^r)c(\Delta^m). \end{aligned}$$

Thus induction on  $r$  proves the proposition.  $\square$

We also have the following helpful result.

**Proposition 3.3.** *Suppose that  $F$  is a degree 2 Siegel modular form of weight  $k$ , level  $\mathcal{N}$ , character  $\chi$ , and lattice coefficients  $c(\Lambda)$ ; recall that  $c(\Lambda) = a(T_\Lambda)$  where  $\Lambda \simeq 2T_\Lambda$ . Fix a prime  $p$  and  $r \geq 1$ ; take  $\Delta \simeq 2I$  relative to a  $\mathbb{Z}$ -basis  $\{x, y\}$ . Set  $\epsilon = 1 + \chi(-1)(-1)^k$ . Then with  $\eta(p)$  as defined in Lemma 3.1 and  $\eta(p^{r+1})$  as defined in Proposition 3.2, we have*

$$\begin{aligned} \eta(p)a(p^r I) - \eta(p^{r+1})a(I) \\ = -\epsilon a \begin{pmatrix} p^{r-1}m & \\ & p^{r+1}m \end{pmatrix} - \epsilon \sum_{\substack{1 \leq u < p/2 \\ u^2 \not\equiv -1 \pmod{p}}} a \left( p^r m \begin{pmatrix} (1+u^2)/p & u \\ u & p \end{pmatrix} \right). \end{aligned}$$

*Proof.* By Proposition 3.2,  $\eta(p^{r+1})c(\Delta) = \sum_{\{\Delta:\Omega\}=(1,p)} c(\Omega^{p^{r-1}})$ . With  $\Omega$  so that  $\{\Delta:\Omega\} = (1,p)$ , we either have  $\Omega = \mathbb{Z}(x+uy) \oplus \mathbb{Z}py$  for  $0 \leq u < p$ , or  $\Omega = \mathbb{Z}px \oplus \mathbb{Z}y$ . Then for  $u \neq 0$ , we have  $\Omega_u = \mathbb{Z}(x+uy) \oplus \mathbb{Z}py \simeq 2p^{r+1} \begin{pmatrix} ((1+u^2)/p & u \\ u & p \end{pmatrix}$ ; from our above discussion on Fourier coefficients of a Siegel modular form  $F$ , we have  $c(\Omega_u^{1/p}) = \chi(-1)(-1)^k c(\Omega_{-u}^{1/p})$ . Similarly,

$$c((\mathbb{Z}px \oplus \mathbb{Z}y)^{1/p}) = \chi(-1)(-1)^k c((\mathbb{Z}x \oplus \mathbb{Z}py)^{1/p}).$$

Further, if  $p$  is odd and  $u^2 \equiv -1 \pmod{p}$ , then by Exercise 5 p. 77 of [4], there is some  $G \in GL_2(\mathbb{Z})$  so that

$${}^t G \begin{pmatrix} (1+u^2)/p & u \\ u & p \end{pmatrix} G = I;$$

hence with  $G' = \text{diag}(-1, 1)G$ , we get

$${}^t G' \begin{pmatrix} (1+u^2)/p & -u \\ -u & p \end{pmatrix} G' = I,$$

and thus  $c(\Omega_u^{1/p}) + c(\Omega_{-u}^{1/p}) = (1 + \chi(-1)(-1)^k)c(\Delta^{p^r})$ . Similarly, when  $p = 2$ ,  $\Omega_1 \simeq 2^{r+2}m \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ , which can be diagonalized using the matrix  $G = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ , and so  $c(\Omega_1^{1/2}) = c(\Delta^{p^r})$ . Using the definition of  $\eta(p)$ , the proposition now follows.  $\square$



Theorem 1.1 is now easy to prove. Take  $\Delta \simeq 2I$ ; recall that  $c(\Delta^{p^r m}) = a(p^r m I)$ . The first claim of (a) follows immediately from Theorem 2.1 and Lemma 3.1. To prove the second claim in (a), we first use Theorem 2.1 to get

$$(3) \quad \tilde{\lambda}_1(p^2)c(\Delta^m) = \chi(p)p^{k-2}\alpha(\Delta; p)c(\Delta^m) + \sum_{\{\Delta:\Omega\}=(1,p)} c(\Omega)$$

and

$$(4) \quad \lambda(p)c(\Delta^{pm}) = \chi(p^2)p^{2k-3}c(\Delta) + \chi(p)p^{k-2} \sum_{\{\Delta:\Omega\}=(1,p)} c(\Omega) + c(\Delta^{p^2}).$$

Solving Equation (4) for the sum on  $\Omega$  and substituting into  $\chi(p)p^{k-2}$ ·Equation (3) yields the second claim in (a).

To prove (b), we first use Theorem 2.1 and Proposition 3.2 to obtain

$$\begin{aligned} a(p^{r+1}I) &= \lambda(p)a(p^r I) - \chi(p^2)p^{2k-3}a(p^{r-1}I) \\ &\quad - \chi(p)p^{k-2}\eta(p^{r+1})a(I). \end{aligned}$$

Next we multiply this equation by  $a(mI)$ , use Theorem 1.1(a) to substitute for  $\lambda(p)a(mI)$ , and use Proposition 3.3 to substitute for  $\eta(p)a(p^r I) - \eta(p^{r-1}I)a(I)$ ; (b) now immediately follows.

For (c), suppose that  $n = p_1^{e_1} \cdots p_t^{e_t}$  where  $p_1, \dots, p_t$  are distinct primes so that  $F$  is an eigenform for  $T(p_i)$  and  $\tilde{T}_1(p_i^2)$  ( $1 \leq i \leq t$ ). For any  $m' \in \mathbb{Z}_+$  with  $(n, m') = 1$ , repeated applications of Proposition 3.2 gives us

$$a(m'nI) = \kappa(p_1^{e_1}) \cdots \kappa(p_t^{e_t})a(m'I).$$

Thus (taking  $m' = m$ ) we have  $a(mnI) = 0$  if  $a(mI) = 0$ . Further (taking  $m' = 1$ ), we have

$$a(nI) = \kappa(p_1^{e_1}) \cdots \kappa(p_t^{e_t})a(I)$$

and hence  $a(I)a(mnI) = a(mI)\kappa(p_1^{e_1}) \cdots \kappa(p_t^{e_t})a(I) = a(mI)a(nI)$ .

#### 4. PROOF OF THEOREM 1.2

As previously noted, the key to proving Theorem 1.1 is Lemma 3.1. We can extend this lemma to some extent, as follows.

**Lemma 4.1.** *Suppose that  $F$  is a degree 2 Siegel modular form of weight  $k$ , level  $\mathcal{N}$ , and character  $\chi$ , and let  $c(\Lambda)$  denote the  $\Lambda$ th coefficient of  $F$ . Suppose that  $p$  is an odd prime and  $\Delta \simeq 2 \begin{pmatrix} 1 & \\ & p \end{pmatrix}$ . For  $m \in \mathbb{Z}_+$  with  $p \nmid m$ , we have*

$$\sum_{\{\Delta:\Omega\}=(1/p,1)} c(\Omega^{pm}) = \chi(-1)(-1)^k c(\Delta^m).$$

*For  $q$  an odd prime with  $\left(\frac{-p}{q}\right) = -1$  and  $q \nmid m$ , we have*

$$\sum_{\{\Delta:\Omega\}=(1/q,1)} c(\Omega^{qm}) = 0.$$

*Proof.* Let  $\{x, y\}$  be a  $\mathbb{Z}$ -basis for  $\Delta$  relative to which  $\Delta \simeq \begin{pmatrix} 2 & \\ & 2p \end{pmatrix}$ . Then the only lattice  $\Omega$  so that  $\{\Delta : \Omega\} = (1, p)$  and  $\Omega^{m/p}$  is even integral is

$$\Omega = \mathbb{Z}px \oplus \mathbb{Z}y \simeq 2p \begin{pmatrix} p & \\ & 1 \end{pmatrix} = 2p \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \\ & p \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

With  $\gamma = \text{diag} \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$ , we have  $F|\gamma = \chi(-1)F$  and consequently  $c \left( 2m \begin{pmatrix} p & \\ & 1 \end{pmatrix} \right) = \chi(-1)(-1)^k c \left( 2m \begin{pmatrix} 1 & \\ & p \end{pmatrix} \right)$ . Hence

$$\sum_{\{\Delta:\Omega\}=(1/p,1)} c(\Omega^{pm}) = \sum_{\{\Delta:\Omega\}=(1,p)} c(\Omega^{m/p}) = \chi(-1)(-1)^k c(\Delta^m).$$

With  $q$  an odd prime with  $\left(\frac{-p}{q}\right) = -1$  and  $q \nmid m$ , there is no lattice  $\Omega$  so that  $\{\Delta : \Omega\} = (1, q)$  and  $\Omega^{m/q}$  is even integral, and hence

$$\sum_{\{\Delta:\Omega\}=(1/q,1)} c(\Omega^{qm}) = 0.$$

□

To prove Theorem 1.2, we begin by making the following definitions. Set  $\eta(1) = 0$ ,  $\kappa(1) = 1$ . For  $q \in \mathcal{S}$  (as defined in the statement of Theorem 1.2), define  $\eta(q)$  as in Lemma 4.1, and set  $\kappa(q) = \lambda(q) - \chi(q)q^{k-2}\eta(q)$ . For  $r \geq 2$ , we define  $\eta(q^r)$  and  $\kappa(q^r)$  using the inductive formulas from Proposition 3.2 (so  $\eta(q^r), \kappa(q^r)$  are determined by  $\eta(q)$ ,  $\lambda(q)$  and  $\tilde{\lambda}_1(q^2)$ ). Then mimicking the proofs of Proposition 3.2 and Theorem 1.1(c) easily yields Theorem 1.2.

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